Answers to test yourself questions

Option A

A1 The beginnings of relativity

1 No, some things may be "relative" in relativity but not all is relative. For one thing, a rotating earth means that it is not in an inertial frame.

2 The acceleration must be very small so as to be negligible. The acceleration due to the rotation on its axis is $a = 0.034\ \text{m s}^{-2}$ and that due to the rotation around the sun is $a = 5.95 \times 10^{-3}\ \text{m s}^{-2}$. In addition, to really consider an observer on earth as being in a true inertial frame there can be no gravity and this can happen only in a frame of reference that is freely falling above the surface of the earth. In such a frame, there is no gravity.

3 You can think of many such experiments. One is to let a ball drop from rest. The ball will fall vertically down (as far as you are concerned) in exactly the same way as if the train were at rest.

4 No we cannot, since the galvanometer would show the same current irrespective of whether it is the coil or the magnet that moves with respect to the ground.

5 You can hang a pendulum from the ceiling. If the train accelerates, the string will not be vertical. If the string is displaced in a given direction, the direction of acceleration will be opposite to that direction.

6 The surface of water in a bucket in a rotating frame of reference would not be flat.

7 If one inertial observer measures that there is a force, and hence acceleration, other inertial observers must agree. According to the observer moving along with the proton, the proton is at rest so it cannot have a magnetic force on it ($F = qvB$ and $v = 0$). So the electric charges in the wire must exert an electric force that is equal to the magnetic force (in magnitude and direction) it experiences according to an observer at rest with respect to the wire.

8 **a** $x' = x - vt$

$= 20 - 15 \times 5.0$

$= -55\ \text{m}$

Time is absolute in pre-Einstein physics and so $t' = t = 5.0$ s.

b $u = u' + v = 5.0 + 15 = 20\ \text{m s}^{-1}$

9 **a** $x = x' + vt$

$= 24 + (-25) \times 5.0$

$= -101\ \text{m}$

Time is absolute in pre-Einstein physics and so $t' = t = 5.0$ s.

b $u' = u - v = -15 - (-25) = 10\ \text{m s}^{-1}$

A2 The Lorentz transformations

10 **a** The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.90^2}} = 2.294$. The time interval of 5.0 min is a proper time interval for the earth observer and so for the Zenga invader the time interval is $\gamma \times 5.0 = 2.294 \times 5.0 = 11.47 \approx 11$ min.

b 11 minutes, by exactly the same argument as in **a**.

11 The length of the cube in the direction of motion is contracted and so the volume of the cube decreases. The density therefore increases. The density will be $\gamma\rho$.

12 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.95^2}} = 3.20$. The train observers measure the proper time interval. So the ground observers measure $\gamma \times 1.0 = 3.20 \times 1.0 = 3.2$ s.

13 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.95^2}} = 3.20$. The 100 m is the length contracted distance and so the length at rest (the proper length) is $100\gamma = 3.20 \times 100 = 320\,\text{m}$.

14 **a** The gamma factor is $\gamma = \dfrac{30}{28} = 1.07$. The speed is then

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{1-\frac{1}{1.07^2}} = 0.36.$$

b Since the trains are identical the proper length of train A is 30 m, which B will measure to be length contracted to 28 m.

c Obviously, 30 m.

15 **a** The interval of 5.0×10^{-8} s is a proper time interval. The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.95^2}} = 3.20$ so the time interval for the observer in the lab is $\gamma \times 5.0 \times 10^{-8} = 3.20 \times 5.0 \times 10^{-8} = 1.6 \times 10^{-7}$ s.

b The distance traveled is $vt = 0.95 \times 3.0 \times 10^8 \times 1.6 \times 10^{-7} = 45.6 \approx 46$ m.

16 **a** The time is $\dfrac{x}{v} = \dfrac{50\,\text{ly}}{0.995c} \approx 50.3$ yr.

b The time taken according to the spacecraft clocks will be the proper time and this is $\dfrac{50.3}{\gamma}$ yr. The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.995^2}} \approx 10$. Hence the time is $\dfrac{50.3}{10} = 5.03$ yr. The students are just over 23 years old when they get to Vega.

17 **a** The time interval of 4.0 years is the proper time interval for the rocket observers for the events: spacecraft leaves earth and spacecraft sends signal. The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.60^2}} = 1.25$. Hence according to the earth clocks the signal is sent after $\gamma \times 4.0 = 1.25 \times 4.0 = 5.0$ yr. During this time the spacecraft has traveled a distance $x = vt = 0.60c \times 5.0 = 3.0$ ly according to earth. This distance will be covered at the speed of light by the signal and so it will arrive on earth after 3.0 years according to earth.

b For the rocket, the earth is at a distance of $x' = vt' = 0.60c \times 4.0 = 2.40$ ly. In the time T it takes the signal to get to earth the earth moved away a distance of $x = vt = 0.60c \times T$ so $cT = 0.60c \times T + 2.4 \Rightarrow T = \dfrac{2.4}{0.4} = 6.0\,\text{yr}$.

18 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.80^2}} = \dfrac{5}{3}$.

a The distance of 8.0 ly is covered at the speed of light and so takes 8.0 years according to earth.

b The distance separating the spacecraft from the space station is $\dfrac{8.0}{5/3} = 4.8\,\text{ly}$ according to the spacecraft. Therefore the space station covers this distance in $\dfrac{4.8\,\text{ly}}{0.80c} = 6.0\,\text{yr}$.

c The spacecraft is 8.0 ly away from earth (according to earth) when the signal is emitted. In the 8.0 years it takes the signal to arrive at earth the spacecraft moved an additional distance of $0.80c \times 8.0 = 6.4\,\text{ly}$. The reply signal will cover a distance cT where T is the required arrival time. Then (since spacecraft will travel a distance of $0.80cT$ in the meantime) $cT = 8.0 + 6.4 + 0.80cT \Rightarrow T = 72\,\text{yr}$.

d The time from the emission of the signal by the spacecraft and its reception is a proper time interval for the spacecraft. According to earth the time interval between these two events is $8.0 + 72 = 80\,\text{yr}$. Hence the time for the spacecraft is $\dfrac{80}{5/3} = 48\,\text{yr}$.

19 **a** According to the ground the light signal will take time T. In this time the rocket will move a distance vT closer to the mirror. Hence, $cT = D + (D - vT) \Rightarrow T = \frac{2D}{c+v} = \frac{4.8\times10^{12}}{1.90\times3.0\times10^{8}} = 8.42\times10^{3} \approx 8.4\times10^{3}$ s.

b The time for the rocket is the proper time interval since the signal is emitted and received at the same place. The gamma factor is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.90^2}} = 2.29$. Hence $T' = \frac{8.42\times10^{3}}{2.29} = 3.7\times10^{3}$ s.

20 **a** $u' = \frac{u-v}{1-\frac{uv}{c^2}}$, $v = 0.6c$, $u = 0.8c$. Then $u' = \frac{0.2c}{1-0.8\times0.6} = 0.385c$.

b The answer is obviously $-0.385c$ but we can verify this from: $u' = \frac{u-v}{1-\frac{uv}{c^2}}$ where now $v = 0.8c$, $u = 0.6c$ so that $u' = \frac{-0.2c}{1-0.8\times0.6} = -0.385c$.

21 **a** $u' = \frac{u-v}{1-\frac{uv}{c^2}}$, $v = -0.6c$, $u = 0.8c$. Then $u' = \frac{-1.40c}{1-0.8\times(-0.6)} = -0.946c$.

b The answer is obviously $0.946c$ but we can verify this from: $u' = \frac{u-v}{1-\frac{uv}{c^2}}$ where now $v = 0.8c$, $u = -0.6c$ so that $u' = \frac{1.40c}{1-(-0.6)\times0.8} = 0.946c$.

22 Here we need to use $u = \frac{u'+v}{1+\frac{u'v}{c^2}}$ with $v = 0.60c$ and $u' = 0.70c$. This gives $u = \frac{0.70c+0.60c}{1+0.70\times0.60} = 0.915c$.

23 Here we need to use $u = \frac{u'+v}{1+\frac{u'v}{c^2}}$ with $v = -0.60c$ and $u' = 0.70c$. This gives $u = \frac{0.70c+(-0.60c)}{1+0.70\times(-0.60)} = 0.172c$.

24 **a** The lifetime is $t = \frac{x}{v} = \frac{2.00\times10^{3}}{0.95\times3.0\times10^{8}} = 7.02\times10^{-6}$ s.

b This observers measures the proper time interval between the events muon created and muon decays and so $\tau = \frac{t}{\gamma} = \frac{7.02\times10^{-6}}{\frac{1}{\sqrt{1-0.95^2}}} = 2.19\times10^{-6}$ s.

25 The lifetime of the pion according to the lab is $t = \frac{20}{v}$ and also $t = \tau\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\times2.6\times10^{-8}$.

Hence, $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\times2.6\times10^{-8} = \frac{20}{v}$. This is best solved on the SOLVER of your GDC. Otherwise,

$v^2\times(2.6\times10^{-8})^2 = 20^2\left(1-\frac{v^2}{c^2}\right)$. This means $\frac{v^2}{c^2}\times60.84 = 400\left(1-\frac{v^2}{c^2}\right) \Rightarrow \frac{v^2}{c^2} = \frac{400}{460.84} \Rightarrow \frac{v}{c} = 0.931$ so that finally $v = 2.8\times10^{8}$ m s^{-1}.

Note: In the following questions the frames S and S′ have their usual meaning i.e. S′ moves past S with velocity v and when the origins coincide clocks are set to zero.

26 $x' = \gamma(x - vt)$ and $t' = \gamma\left(t - \frac{vx}{c^2}\right)$; the gamma factor is $\gamma = \frac{1}{\sqrt{1-0.75^2}} = 1.5119$. Hence

$x' = 1.5119 \times (600 - 0.75 \times 3 \times 10^8 \times 2.0 \times 10^{-6}) = 226.8 \approx 230$ m and

$t' = 1.5119 \times \left(2.0 \times 10^{-6} - \frac{0.75 \times 3 \times 10^8 \times 600}{(3 \times 10^8)^2}\right) = 7.6 \times 10^{-7}$ s.

27 $x = \gamma(x' + vt')$ and $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$; the gamma factor is $\gamma = \frac{1}{\sqrt{1-0.98^2}} = 5.0252$. Hence

$x = 5.0252 \times (0 + 0.98 \times 3 \times 10^8 \times 6.0 \times 10^{-6}) = 8.9 \times 10^3$ m and

$t = 5.0252 \times (6.0 \times 10^{-6} - 0) = 3.0 \times 10^{-5}$ s.

28 The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.60^2}} = 1.25$. The S clocks read $t = \frac{120}{0.60 \times 3 \times 10^8} = 6.667 \times 10^{-7}$ s when S′ passes the 120 m mark in S. Hence with $t' = \gamma\left(t - \frac{vx}{c^2}\right)$ we find

$t' = 1.25 \times \left(6.667 \times 10^{-7} - \frac{0.60 \times 3 \times 10^8 \times 120}{(3 \times 10^8)^2}\right) = 5.3 \times 10^{-7}$ s.

29 a The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.60^2}} = 1.25$.

i $\Delta x' = \gamma(\Delta x - v\Delta t) = 1.25 \times (1200 - 0.600 \times 3 \times 10^8 \times 6.00 \times 10^{-6}) = 150$ m

$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 1.25 \times \left(6.00 \times 10^{-6} - \frac{0.600 \times 3 \times 10^8 \times 1200}{(3 \times 10^8)^2}\right) = 4.5 \times 10^{-6}$ s

ii The two events are separated by a distance of 1200 m and a time interval of 6.0 μs. A signal from event 1 to event 2 would take $\frac{1200}{6.0 \times 10^{-6}} = 2.0 \times 10^8$ m s^{-1} which is less than the speed of light so event 1 could cause event 2.

b $\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 0$ implies $6.0 \times 10^{-6} - \frac{1200v}{c^2} = 0 \Rightarrow v = \frac{6.0 \times 10^{-6} \times 3 \times 10^8}{1200}c = 1.5c$ which is impossible.

30 a The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.60^2}} = 1.25$.

i $\Delta x' = \gamma(\Delta x - v\Delta t) = 1.25 \times (1200 - 0.600 \times 3 \times 10^8 \times 3.00 \times 10^{-6}) = 825$ m

$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 1.25 \times \left(3.00 \times 10^{-6} - \frac{0.600 \times 3 \times 10^8 \times 1200}{(3 \times 10^8)^2}\right) = 7.5 \times 10^{-7}$ s

ii The two events are separated by a distance of 1200 m and a time interval of 3.0 μs. A signal from event 1 to event 2 would take $\frac{1200}{3.0 \times 10^{-6}} = 4.0 \times 10^8$ m s^{-1} which is more than the speed of light so event 1 could not cause event 2.

b $\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) < 0$ implies $3.0 \times 10^{-6} - \frac{1200v}{c^2} < 0 \Rightarrow v > \frac{3.0 \times 10^{-6} \times 3 \times 10^8}{1200}c > 0.75c$. This says that it possible to find a frame in which event 2 occurs **before** event 1. In the frame S event 1 occurs before event 2. But this is not a problem since event 1 is not the cause of event 2.

31 The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.60^2}} = 1.25$,

$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 1.25 \times \left(0 - \frac{0.600 \times 3 \times 10^8 \times 1200}{(3 \times 10^8)^2}\right) = -3.0 \times 10^{-6}$ s and so event 1 occurs first

A3 Spacetime diagrams

Note: In the questions that follow the space-time diagrams represent two inertial frames. The black axes represent frame S. The red axes represent a frame S′ that moves past frame S with velocity v.

32 Using the dashed line and $v = \frac{x}{t} = \frac{x}{ct}c = \frac{0.6}{1.0}c = 0.6c$.

33 Using the dashed line and $v = \frac{x}{t} = \frac{x}{ct}c = \frac{-0.8}{1.0}c = -0.8c$.

34 **a** **i** Blue line

ii Green line

b We must draw the worldline of a photon starting at $x = 1.0$ m and $t = 0$ and see where the worldline intersects the two time axes.

i $ct = 0.61\text{ m} \Rightarrow t = 2.0 \times 10^{-9}$ s in S

ii $ct' \approx 0.50\text{ m} \Rightarrow t' = 1.7 \times 10^{-9}$ s in S′

35 **a** Using the dashed line and $v = \frac{x}{t} = \frac{x}{ct}c = \frac{1.0}{2.0}c = 0.50c$.

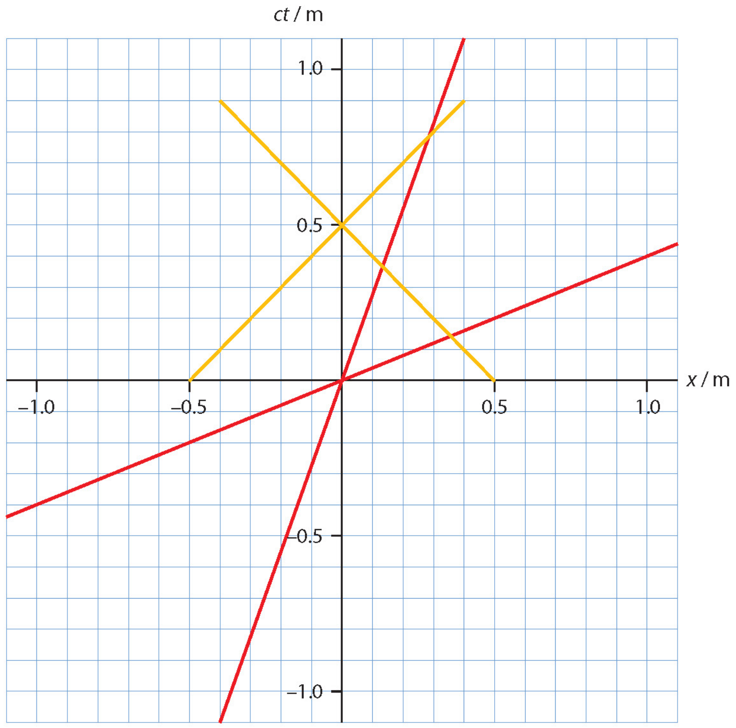
b From the diagram

i $ct = 2.0\text{ ly} \Rightarrow t = 2.0$ y in S

ii $ct' \approx 1.8\text{ ly} \Rightarrow t' = 1.8$ y in S′

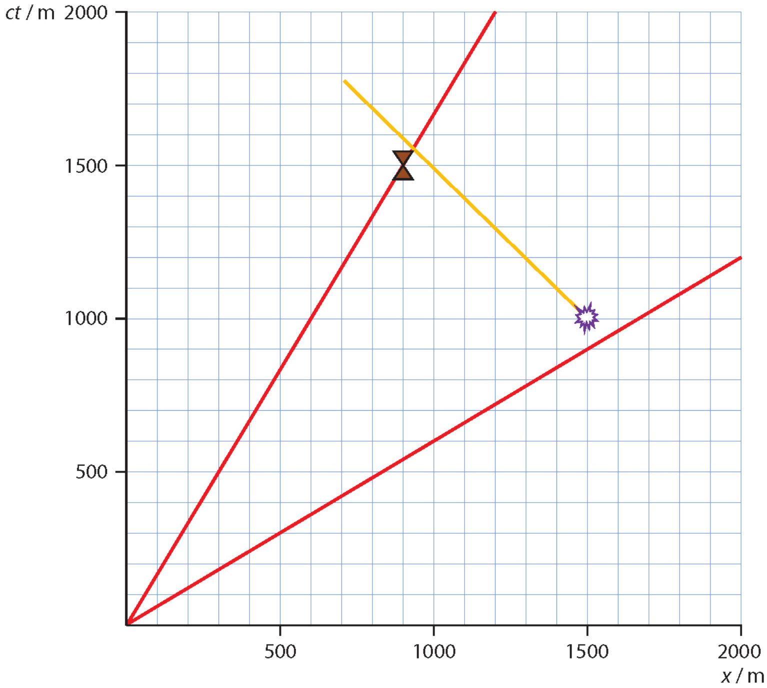
36 **a** We draw lines from the events parallel to the primed space axis to see that the lamp at $x = +\,0.5$ m turns on first.

b The lines are 45 lines as shown on the diagram.



c Light from the lamp at $x = +\,0.5$ m reaches the observer in S′ first.

37 a The photon wordline shows that photons arrive after the shield is put on and so the spacecraft is safe.



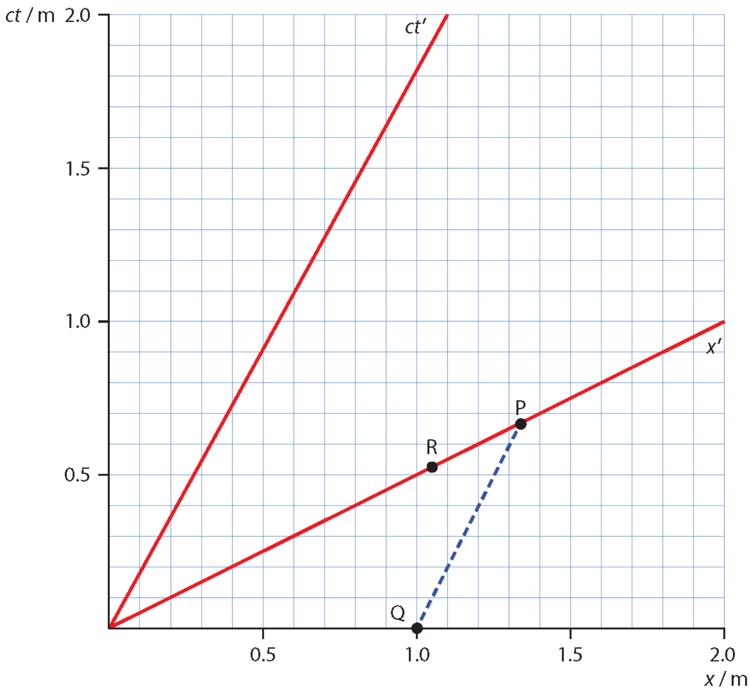
b In the S frame we know that $x = 1500$ m, $ct = 1000$ m. The speed of the spacecraft is 0.60c and so the gamma factor is 1.25. Therefore:

i $ct' = c\gamma\left(t - \frac{vx}{c^2}\right) = \gamma\left(ct - \frac{vx}{c}\right) = 1.25 \times \left(1000 - \frac{0.60c \times 1500}{c}\right) = 125\ \text{m}$. Hence $t' = \frac{125}{3 \times 10^8} = 0.42\,\mu\text{s}$

ii $x' = \gamma(x - vt) = 1.25 \times (1500 - 0.60 \times 1000) = 1125\text{m}$

38 a The speed of the primed frame is 0.50 c. The gamma factor is then $\gamma = \frac{1}{\sqrt{1 - 0.50^2}} = 1.1547 \approx 1.2$.

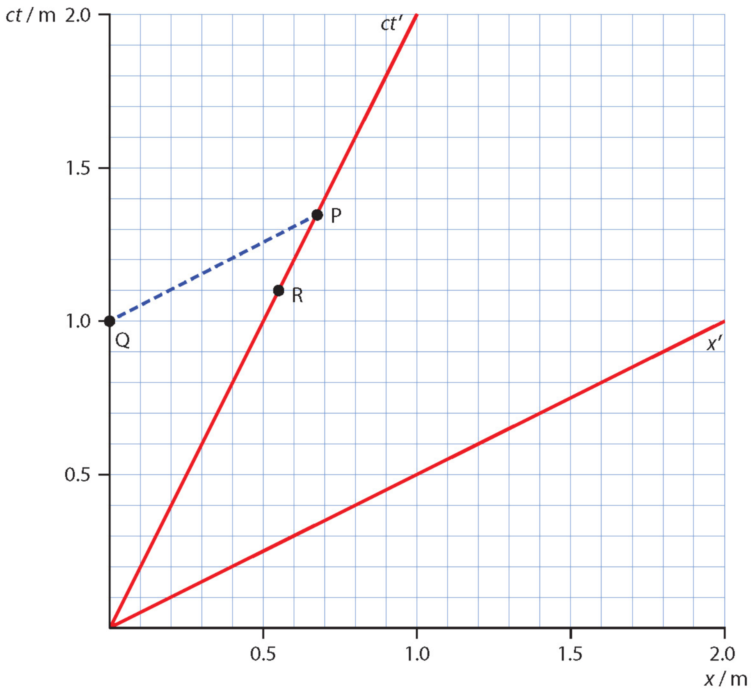
Event P has the same x' coordinate as event Q. The coordinates of Q in S are $x = 1$, $ct = 0$. Hence $x' = \gamma(x - vt) = 1.2 \times (1 - 0) = 1.2$ m. The time coordinate of P is clearly zero. Knowing that the space coordinate of P is about 1.2 m we can measure along the primed x axis to estimate the position of the point with $x' = 1$ m is approximately at point R.



b For the general case again consider event Q that has coordinates in S of $x = 1\text{ m}$ and $t = 0$. Then $x' = \gamma(x - vt) = \gamma(1-0) = \gamma$. $(x', ct') = (\gamma, 0)$

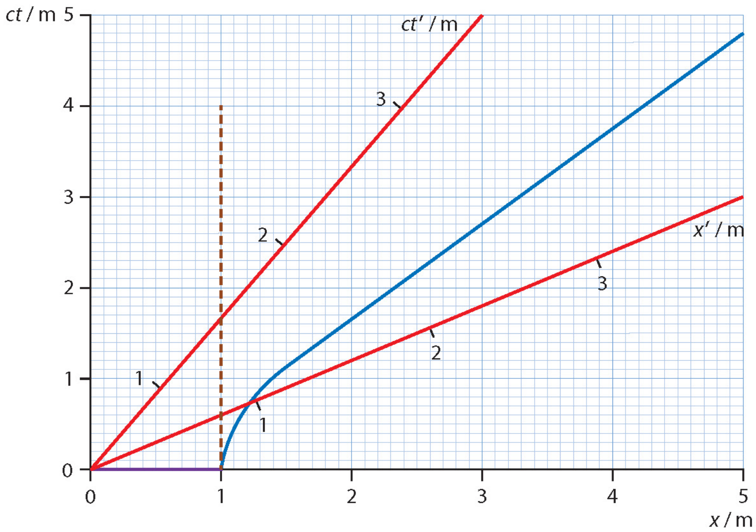
39 a The speed of the primed frame is 0.50 c. The gamma factor is then $\gamma = \dfrac{1}{\sqrt{1-0.50^2}} = 1.1547 \approx 1.2$.

Event P has the same t' coordinate as event Q. The coordinates of Q in S are $x = 0$, $ct = 1\text{ m}$. Hence $ct' = c\gamma\left(t - \dfrac{vx}{c^2}\right) = 1.2 \times (1-0) = 1.2\text{ m}$. The space coordinate of P is clearly zero. Knowing that the time coordinate of P is about 1.2 m we can measure along the primed t axis to estimate the position of the point with $ct' = 1\text{ m}$ is approximately at point R.



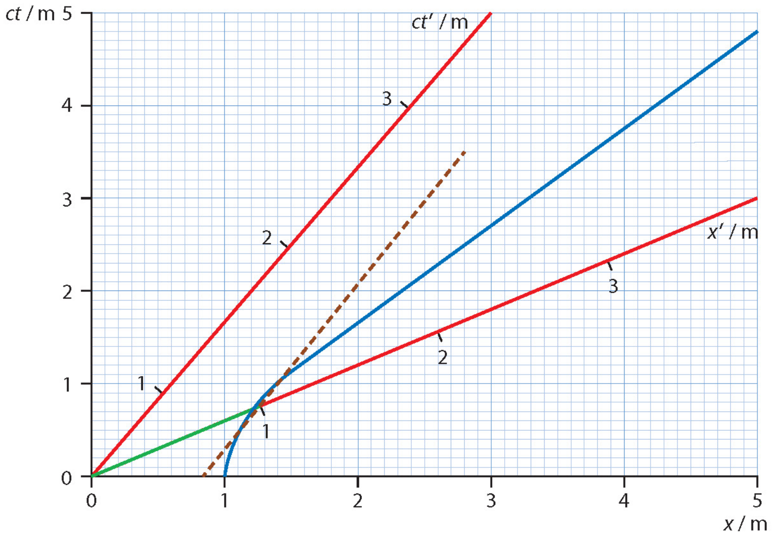
b For the general case again consider event Q that has coordinates in S of $x = 0$ and $ct = 1\text{ m}$. Then $ct' = c\gamma\left(t - \dfrac{vx}{c^2}\right) = \gamma\left(ct - \dfrac{vx}{c^2}\right) = \gamma(1-0) = \gamma$. P has coordinates $(x', ct') = (0, \gamma)$.

40 a The dotted line is the worldline of the right end of the rod. It intersects the primed x axis at a point that is less than 1 m.



b i See green line segment

ii Dotted line intersects the x axis at a point that is less than 1 m.



41 Draw the dotted line which intersects the time axis at $ct = 1.25$ m. Hence $t = 4.2$ ns.

A4 Relativistic mechanics (HL)

42 a The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.50^2}} = 1.1547$ and so the kinetic energy (and the energy that needs to be supplied) is $E_K = (\gamma - 1)mc^2 = (1.1547 - 1) \times 0.511 = 0.079$ MeV.

b The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.90^2}} = 2.2942$ and so the kinetic energy (and the energy that needs to be supplied) is $E_K = (\gamma - 1)mc^2 = (2.2942 - 1) \times 0.511 = 0.66$ MeV.

c The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.99^2}} = 7.0888$ and so the kinetic energy (and the energy that needs to be supplied) is $E_K = (\gamma - 1)mc^2 = (7.0888 - 1) \times 0.511 = 3.1$ MeV.

43 $E_K = (\gamma - 1)m_0c^2 = 10m_0c^2 \Rightarrow \gamma = 11$. Hence, $11 = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow 1 - \dfrac{v^2}{c^2} = \dfrac{1}{11^2} \Rightarrow \dfrac{v}{c} = \sqrt{1-\dfrac{1}{11^2}} = 0.996.$

44 $E_T = 5mc^2 = 5 \times 938 = 4690$ MeV. From $E^2 = (mc^2)^2 + p^2c^2$, $pc = \sqrt{E^2 - (mc^2)^2} = \sqrt{4690^2 - 938^2} = 4595$ MeV. Hence $p = 4.6 \times 10^3$ MeV c^{-1}.

45 The momentum is huge and so the total energy will be just 350 MeV. Explicitly,
$E = \sqrt{(mc^2)^2 + p^2c^2} = \sqrt{0.511^2 + 350^2} \approx 350$ MeV.

46 $p = 685 \text{ MeV c}^{-1} = \dfrac{685 \times 10^6 \times 1.6 \times 10^{-19}}{3.0 \times 10^8} = 3.65 \times 10^{-19}$ N s

47 The total energy is $E = \sqrt{(mc^2)^2 + p^2c^2} = \sqrt{938^2 + 500^2} = 1063$ MeV.
Hence $E_K = 1063 - 938 = 125$ MeV.

48 The gamma factor is $\gamma = \frac{200}{135} = 1.4815$. The speed is then

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$1-\frac{v^2}{c^2} = \frac{1}{1.4815^2}$$

$$\frac{v^2}{c^2} = 0.54438$$

$$v = 0.738c$$

49 The total energy will be $E = \sqrt{(mc^2)^2 + p^2c^2} = \sqrt{(0.938)^2 + 1200^2} \approx 1200$ GeV. The kinetic energy is $E_K = 1200 - 0.938 = 1199.1$ GeV. The accelerating voltage to 2 s.f. is then 1200 GV.

50 The gamma factor is $\gamma = \frac{1}{\sqrt{1-0.99^2}} = 7.0888$. The momentum is then

$$p = \gamma mv = \frac{\gamma mc^2 v}{c^2} = \frac{7.0888 \times 938 \times 0.99c}{c^2}\text{MeV}$$

$$= \frac{7.0888 \times 938 \times 0.99}{c}\text{MeV}$$

$$= 6.58 \times 10^3 \text{ MeV c}^{-1}$$

$$\approx 6.6 \times 10^3 \text{ MeV c}^{-1}$$

A faster alternative is: after getting the gamma factor find the total energy to be $E = \gamma mc^2 = 7.0888 \times 938 = 6649$ MeV and then from $E^2 = (mc^2)^2 + p^2c^2 \Rightarrow pc = \sqrt{6649^2 - 0.938^2} \approx 6.6 \times 10^3$ MeV.

51 The energy is $E = \sqrt{m^2c^4 + p^2c^2} = \sqrt{0.938^2 + 1.5^2} = 1.7691$ GeV. The gamma factor is therefore $\gamma = \frac{1.7691}{0.938} = 1.8861$ and the speed is:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$1-\frac{v^2}{c^2} = \frac{1}{1.8861^2}$$

$$\frac{v^2}{c^2} = 0.71889$$

$$v = 0.848c$$

52 **a** The work done is

$W = Fd = eEd = 1.6 \times 10^{-19} \times 5.0 \times 10^6 \times 10^3 = 8.0 \times 10^{-10}\text{ J} = 5.0 \times 10^9\text{ eV} = 5.0 \times 10^3$ MeV. Therefore $E_K = 5.0 \times 10^3$ MeV (or 5.0 GeV).

b The total energy is $E_T = (938 + 5.0 \times 10^3) = 5.938 \times 10^3$ MeV. Hence $\gamma = \frac{E_T}{m_0c^2} = \frac{5.938 \times 10^3}{938} = 6.33$.

Hence the speed is: $6.33 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{6.33^2} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{6.33^2}} = 0.987$.

53 a From $p=\gamma mv$ and $E=\gamma mc^2$ we get by dividing side by side to get rid of the gamma factor: $\frac{p}{E}=\frac{v}{c^2}$.

Hence $v=\frac{pc^2}{E}=\frac{pc^2}{\sqrt{m^2c^4+p^2c^2}}$ as required.

b For the electron with momentum 1.00 MeV c^{-1}, $\sqrt{m^2c^4+p^2c^2}=\sqrt{0.511^2+1.00^2}=1.123$. For the proton, $\sqrt{m^2c^4+p^2c^2}=\sqrt{938^2+1.00^2}\approx 938$. Hence the ratio of the speeds is $\frac{v_e}{v_p}=\frac{938}{1.123}=835$.

c For a momentum of 1.00 GeV c^{-1}, for the electron $\sqrt{m^2c^4+p^2c^2}=\sqrt{0.511^2+(10^3)^2}\approx 10^3$ and for the proton $\sqrt{m^2c^4+p^2c^2}=\sqrt{938^2+(10^3)^2}=1371$ so that $\frac{v_e}{v_p}=\frac{1371}{1000}=1.37$.

d As the momentum increases we may neglect the rest energy in which case both speeds tend to become the speed of light and so the ratio approaches 1.

54 a The momentum of the fragments must be zero since the original momentum before the breakup was zero. The gamma factor is $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-0.85^2}}=1.8983$ and so the momentum

is 1.8983×125 MeV $c^{-2}\times 0.85c=201.7$ MeV c^{-1}. This is also the momentum of the other fragment. Hence the total energy of each of the fragments is $E_1=\sqrt{125^2+201.7^2}=237.3$ MeV and $E_2=\sqrt{250^2+201.7^2}=321.2$ MeV. The heavier fragment has gamma factor given by $E_2=\gamma mc^2=\gamma\times 250$ MeV $=321$ MeV and so $\gamma=1.284$. Hence the speed is:

$$1.284=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow 1-\frac{v^2}{c^2}=\frac{1}{1.284^2}\Rightarrow v=0.627c.$$

b The total energy of the system therefore is $237.3+321.2=558$ MeV. This is the rest energy of the particle that broke up and so its rest mass is 558 MeV c^{-2}.

55 a The total momentum of the electron – positron pair is zero. If only one photon is produced it will have momentum violating the law of momentum conservation.

b Again because of momentum conservation.

c The total energy of the electron is $E=mc^2+E_K=0.51+2.0=2.51$ MeV. The positron has the same energy. The total energy is then $E_T=2\times 2.51\approx 5.0$ MeV. The photons must have the same energy because they move in opposite directions with the same momentum (magnitude) and hence the same wavelength. So each has an energy of about 2.5 MeV.

56 The gamma factor is $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-0.80^2}}=\frac{5}{3}$ and so the total energy

of the pion is $E_T=\frac{5}{3}\times 135=225$ MeV. The momentum of the pion is:

$225=\sqrt{(135)^2+(pc)^2}\Rightarrow pc=\sqrt{225^2-135^2}=180$ MeV $\Rightarrow p=180$ MeV c^{-1}. Conservation of energy and momentum gives:

$$225=hf_A+hf_B$$

$$180=\frac{hf_A}{c}-\frac{hf_B}{c}$$

To simplify things set $c = 1$ which is alright since we are going to take a ratio and units will not be important. Then

$$225 = hf_A + hf_B$$

$$180 = hf_A - hf_B$$

Adding, $f_A = \frac{405}{2h}$, subtracting, $f_B = \frac{45}{2h}$. The ratio is therefore $\frac{f_A}{f_B} = \frac{405}{45} = 9.0$.

57 The momentum of the two bodies is zero and so the particle they form is produced at rest. The gamma factor is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.80^2}} = \frac{5}{3}$ and so the momentum of each particle is $\frac{5}{3} \times 3.0 \times 0.80 \times 3 \times 10^8 = 1.2 \times 10^9$ N s.

Hence the total energy of each of the particles is $E = \sqrt{(3.0 \times 9.0 \times 10^{16})^2 + (1.2 \times 10^9 \times 3 \times 10^8)^2} = 4.5 \times 10^{17}$ J.

The rest energy of the particle that is formed and so its rest mass is therefore $2 \times 4.5 \times 10^{17} = 9.0 \times 10^{17}$ J and hence the rest mass is $\frac{9.0 \times 10^{17}}{9.0 \times 10^{16}} = 10$ kg.

58 **a** $p = \gamma m_0 v$

b $E = \gamma m_0 c^2$

c Eliminating the gamma factor from **a** and **b** we get $\frac{p}{E} = \frac{v}{c^2}$. Hence $v = \frac{pc^2}{E}$ as required.

d For a massless particle, $E = \sqrt{0 + p^2c^2} = pc$ hence $v = \frac{pc^2}{E} = \frac{pc^2}{pc} = c$.

59 **a** The work done on the particle is qV and this goes into increasing the kinetic energy of the particle. I.e. $qV = (\gamma - 1)mc^2$ from which the result follows: $\gamma - 1 = \frac{qV}{mc^2} \Rightarrow \gamma = 1 + \frac{qV}{mc^2}$.

b The gamma factor is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.998^2}} = 15.819$. Hence from (a) $15.819 = 1 + \frac{1 \times V}{938} \Rightarrow V = 13.9$ GV

60 Let the electron be at rest and assume that a photon of energy E is absorbed by the electron. The momentum of the photon is $p = \frac{E}{c}$ and this will be the momentum of the electron after absorption. The total energy of the electron after absorption is thus $\sqrt{(mc^2)^2 + p^2c^2} = \sqrt{(mc^2)^2 + E^2}$. By conservation of energy we must also have that this total energy equals the total energy of the system before absorption which is $mc^2 + E$. Therefore: $mc^2 + E = \sqrt{(mc^2)^2 + E^2}$. Squaring gives

$$(mc^2)^2 + 2Emc^2 + E^2 = (mc^2)^2 + E^2$$

$$2Emc^2 = 0$$

$$E = 0$$

which is impossible: a photon cannot have zero energy. Therefore the assumption that the electron could absorb the photon is false. The case of emitting the photon is similar.

A5 General relativity (HL)

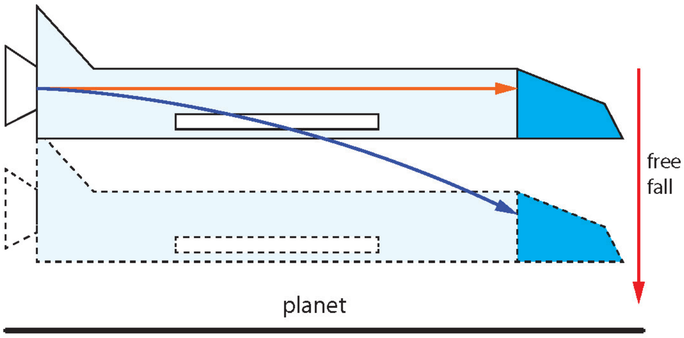
61 This is a true statement. Light follows the geodesics of spacetime. In the absence of matter, the spacetime is flat and the geodesics are the ordinary straight lines. In a curved spacetime the "straight lines" i.e. the geodesics look bent because we are prejudiced into thinking in terms of flat spacetimes.

62 By the equivalence principle, this frame of reference is equivalent to one which is at rest in a gravitational field. The gravitational field strength is then directed to the left. The helium balloon will "rise" i.e. move opposite to the gravitational field. I.e. it will move to the right.

63 For exactly the same reasons as in problem **62** the flame will bend to the right.

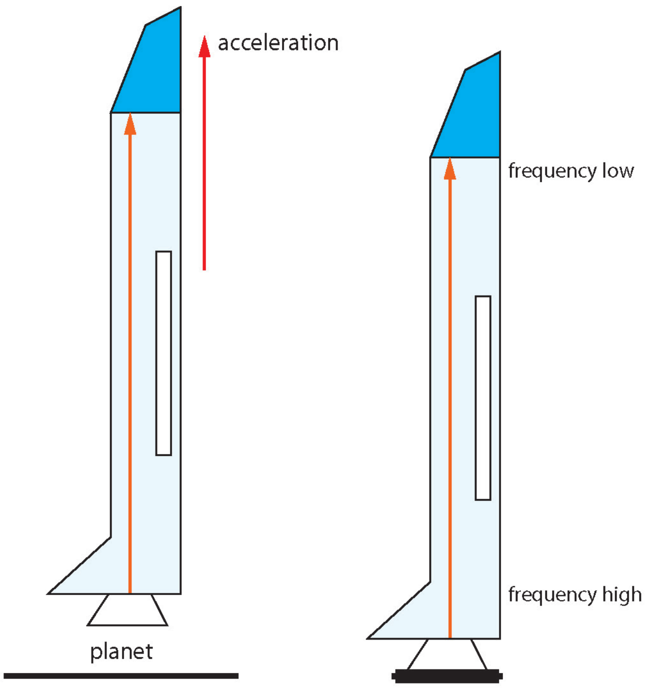
64 **a** The equivalence principle states that inertial effects (i.e. effects due to acceleration) cannot be distinguished from effects of gravitation. More precisely, it states that an accelerating frame of reference in outer space is equivalent to a frame of reference at rest in a uniform gravitational field whose field strength is the same as the acceleration of the other frame. It also states that a freely falling frame of reference in a gravitational field is equivalent to an inertial frame of reference.

b **i** Consider a rocket that is **freely falling** in a gravitational field. According to observers inside the rocket, the ray of light that is emitted from the back wall of the rocket will travel on a straight line and hit the front of the rocket at a point that has the same distance from the floor as the point of emission (path of light shown in the orange line). This is because to the occupants of the rocket, the rocket is equivalent to a truly inertial frame of reference.



But the rocket is seen to be falling by an observer outside. By the time the light ray goes across, the rocket has fallen and so the ray appears to be following the curved path shown in blue. Thus the outside observer claims that **in a gravitational field, light bends** towards the mass causing the field.

ii The diagram on the left shows a rocket accelerating in outer space. The diagram to the right shows a rocket at rest on a massive body which is thus equivalent to the first frame. A ray of light is emitted from the back of the rocket and is received at the front.



To an observer outside the rocket on the left, the front of the rocket is moving away from the light ray and so there should be a Doppler redshift. The observer outside expects that the frequency of light measured at the reception point should be smaller than that at emission. Hence the outside observer must conclude that **as the ray of light moves higher in the gravitational field it suffers a redshift**. But frequency is the number of wavefronts received per second so how can the frequency change? The answer has to be that when one second goes by, according to a clock at the base, more than a second goes by, according to a clock at the top, i.e. the equivalence principle predicts **gravitational time dilation**: the interval of time between two events is longer when measured by a clock far from the gravitational field compared to a clock near the gravitational field.

65 As the radius gets smaller, a time will be reached when the radius of the object becomes equal to the Schwarzschild radius. The bending of space around the object will be substantial and the object will become a black hole.

66 The period of oscillation of the mass at the end of a spring is $T = 2\pi\sqrt{\frac{m}{k}}$. So the acceleration of gravity, i.e. the gravitational field strength does not enter. Hence the period will be the same in **a** and **b**.

67 The emitted frequency is $f = \frac{3.00 \times 10^8}{500.0 \times 10^{-9}} = 6.00 \times 10^{14}$ Hz. The shift is found from

$$\frac{\Delta f}{f} = \frac{gH}{c^2} \Rightarrow \Delta f = 6.00 \times 10^{14} \times \frac{9.81 \times 50.0}{(3.00 \times 10^8)^2} == 3.27 \text{ Hz}.$$

68 a As the signal moves away from the gravitational field of the star the frequency decreases by the gravitational redshift effect. Hence the wavelength increases.

b Time slows down near a collapsed star and so the time in between reception of the signals by the distant spacecraft increases so the frequency of reception decreases.

c For the same reason the duration of the pulses increases.

69 The acceleration experienced by the clock on the circumference will be greater. By the equivalence principle this clock will behave as an identical clock in a gravitational field. It will therefore run slow relative to a clock in a smaller gravitational field.

70 The mass of the earth is $M = 6.0 \times 10^{24}$ kg. The Schwarzschild radius of the earth is

$R = \frac{2GM}{c^2} = \frac{2 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{9.0 \times 10^{16}} = 8.89 \times 10^{-3}$ m. Its density would then be

$\rho = \frac{M}{\frac{4\pi R^3}{3}} = \frac{6.0 \times 10^{24}}{\frac{4\pi(8.89 \times 10^{-3})^3}{3}} = 2.0 \times 10^{30}$ kg m^{-3}. (This is larger than nuclear densities by a factor of 10^{13}.)

71 From $R = \frac{2GM}{c^2}$, $R = \frac{2 \times 6.67 \times 10^{-11} \times 2.0 \times 10^{31}}{(3.0 \times 10^8)^2} = 3.0 \times 10^4$ m.

72 A geodesic is a curve in spacetime which has the least length compared to any other curve with the same beginning and end points. A particle upon which the net force is zero follows a geodesic.

73 a In Newtonian mechanics the path would be explained by saying that a gravitational attractive force acts on the particle changing the original path in a curved path around the massive object.

b In relativity the path is explained by saying that the particle follows the geodesic in the curved spacetime around the massive object.

74 **a** From the point of view of an observer inside the spacecraft the ball will move on a straight line parallel to the floor and therefore hit the opposite wall at the same height as that at the point of launch. (From the point of view of an inertial observer outside with respect to whom the spacecraft moves upwards, the path will also be a straight line with an upward slope.)

b The situation is equivalent to a frame of reference in a gravitational field. Therefore the ball will curve towards the floor following a parabolic path and will hit the opposite wall lower.

75 **a** From the point of view of an inertial observer outside the spacecraft the light ray will travel along the original straight line as before entering the spacecraft hitting the opposite wall of the spacecraft at a point closer to the floor than at the point of entry. From the point of view of an observer inside the spacecraft the ray will also move a long a straight line with a downward slope.

b The frame is now equivalent to a frame of reference at rest on the surface of a massive body. The ray of light will follow a curved path hitting the opposite wall of the spacecraft at a point closer to the floor than at the point of entry and lower than the answer in **a**.

76 **a** Because rays of light coming from low in the sky will be bent, these will never reach the observer. The observer sees that his horizon is rising and he can only see things within a vertical cone whose angle is decreasing.

b Once inside the event horizon the observer can only see rays of light falling "vertically" into the black hole, i.e. only along a single line.

77 The plane is flying at essentially a constant height and so on the surface of a sphere. This is curved and so the plane follows the geodesics of the sphere (these are great circles – circles whose plane goes the center of the earth) because these have the least length so the least amount of fuel is being used.

78 Einstein through the contraption into the air. Being in free fall, the brass ball is effectively weightless since, by the equivalence principle, it is equivalent to a ball in zero gravitational field. The spring will then pull the mass in the bowl. Einstein was very proud of his present and showed it to all who visited him.

79 We have that $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\dfrac{R_S}{r}}}$, i.e. $2.0 = \dfrac{1.0}{\sqrt{1-\dfrac{R_S}{r}}}$ and so $1-\dfrac{R_S}{r} = \dfrac{1}{4}$, $\dfrac{R_S}{r} = \dfrac{3}{4}$ giving finally $r = \dfrac{4}{3}R_S$. The observer is at distance of $r = \dfrac{1}{3}R_S$ from the event horizon.

80 It will take longer since clocks near massive objects run slow compared to clocks far away. The accelerating spacecraft is equivalent to one at rest in a gravitational field.

81 **a** A black hole is a singularity in spacetime, a point of infinite curvature.

b $R = \dfrac{2GM}{c^2} = \dfrac{2\times 6.67\times 10^{-11}\times 5.0\times 10^{35}}{9.0\times 10^{16}} = 7.4\times 10^{8}\text{ m}$.

c This radius is the distance from the black hole where the escape speed is equal to the speed of light. The black hole does not have a radius since it is a point.

d The observer next to the source measures a period of $T = \dfrac{1}{f} = \dfrac{1}{7.50\times 10^{14}} = 1.3\times 10^{-15}$ s.

e The distant observer will measure a period of $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\dfrac{R_S}{r}}} = \dfrac{1.3\times 10^{-15}}{\sqrt{1-\dfrac{R_S}{1.1R_S}}} = 4.4\times 10^{-15}$ s and hence a frequency of $f = \dfrac{1}{4.4\times 10^{-15}} = 2.3\times 10^{14}$ Hz.

82 a $R_S = \dfrac{2GM}{c^2}$

b The area is $A_S = 4\pi R_S^2 = \dfrac{16\pi G^2 M^2}{c^4}$.

c Mass constantly falls into the black hole and so the radius keeps increasing. Hence the area also increases.

d Entropy is another quantity that always increases. The black hole has thermodynamic properties (as discovered independently by D. Christodoulou and J. Bekenstein) and in fact behaves as a real thermodynamic black body that radiates according to the Stefan-Boltzmann law. This is because of quantum effects as explained by S. Hawking. The effective "temperature" of the black hole is inversely proportional to its mass. This means that small black holes radiate a lot.

83 a The ray will fall straight into the hole.

b The rays will be bent and enter the observer's eye.